Uniformly Best Wavenumber Approximations by Spatial Central Difference Operators

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Philosophy...

Consider a central finite difference stencil

$$\frac{\mathrm{d}u}{\mathrm{d}x}\Big|_{x=x_j}\approx\sum_{k=1}^p c_k^{(p)}(u_{j+k}-u_{j-k})$$

Philosophy

Error in numerical solution is governed by the truncation error of the finite difference stencil.

Strategy Choose $c_k^{(p)}$ to eliminate terms up to $\mathcal{O}(\Delta x^{2p})$

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Conclusion

Example - Advection equation

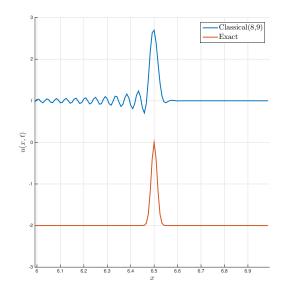
$$u_t + u_x = 0, \quad 0 \le x < 6, \quad t \ge 0$$

 $u(x, 0) = 2 \exp\left(-3200\left(x - \frac{1}{2}\right)^2\right)$

Periodic Boundary Conditions

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Analytic dispersion relation

 $u_t + u_x = 0$

Assume *u* has uniformly convergent Fourier series. Consider general Fourier mode, $u(x, t) = \exp(i(\kappa x - \omega t))$

$$\begin{split} &\omega = \kappa, & \text{Analytic dispersion relation} \\ &v_p = \frac{\omega}{\kappa}, & \text{Phase speed} \\ &v_g = \frac{\mathrm{d}\omega}{\mathrm{d}\kappa}, & \text{Group speed} \end{split}$$

Speeds independent of wavenumber \Rightarrow Non-dispersive solution!

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Numeric dispersion relation

$$(u_t)_j + \sum_{k=1}^p c_k^{(p)} (u_{j+k} - u_{j-k}) = 0$$
$$u_j(t) = \exp(i(\kappa x_j - \bar{\omega} t))$$
$$\Rightarrow \underbrace{\bar{\omega}}_{\xi} \Delta x_j = 2 \sum_{k=1}^p c_k^{(p)} \sin(k \underbrace{\kappa \Delta x}_{\xi})$$

Speeds dependent on wavenumber \Rightarrow Inherently dispersive solution! Incorrect phase speed, $\bar{v}_{p} = \frac{\bar{\omega}}{\kappa}$, and group speed, $\bar{v}_{g} = \frac{d\bar{\omega}}{d\kappa}$

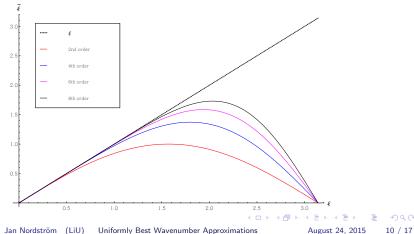
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Dispersion Relations ...

Can show that $\bar{\xi} < \xi$, $\bar{v}_p < v_p$, $\bar{v}_g < v_g$ for all classical stencils. In fact

$$\xi - \bar{\xi} = \frac{2^p}{\binom{2p}{p}} \int_0^{\xi} (1 - \cos\left(\xi'\right))^p \mathrm{d}\xi'$$



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Alternative philosophy

Problems involving high frequencies / wavenumbers that travel for long times have errors dominated by bad numerical dispersion.

- Fluid dynamics
- Aeroacoustics
- Electromagnetism
- Seismology

Philosophy

Consider dispersion error

$$\mathsf{E}(\xi, \mathbf{a}) = \xi - ar{\xi}(\xi, \mathbf{a})$$

when choosing stencil coefficients

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A new problem...

Consider finite difference stencil

$$\frac{\mathrm{d}u}{\mathrm{d}x}\Big|_{x=x_j} = \sum_{k=1}^{p+n} a_k (u_{j+k} - u_{j-k}) + \mathcal{O}(\Delta x^{2p})$$
$$\bar{\xi} = 2\sum_{k=1}^{p+n} a_k \sin(k\xi)$$

- Accuracy constraint Order *O*(Δx^{2p})
- Leaves *n* degrees of freedom
- Use these to minimise dispersion error uniformly, i.e. $\|\xi-ar{\xi}\|_\infty$

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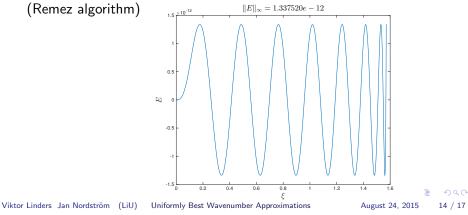


Main results

Theorem

There is a unique stencil **a** that minimises $\|\xi - \overline{\xi}(\xi, \mathbf{a})\|_{\infty}$. The error of this stencil oscillates n + 1 times.

Can device a convergent algorithm for finding best possible **a**



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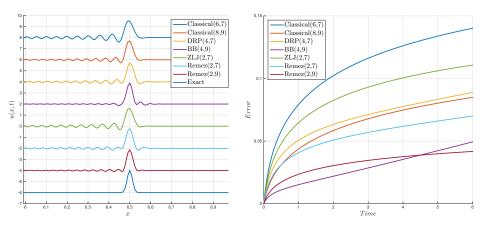
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[1] Tam, Webb (1993)

[2] Zingg, Lomax, Jurgens (1996)[3] Bogey, Bailly (2004)

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Conclusion

- For many problems the numerical error comes from inaccurate approximation of Dispersion Relation
- Classical philosophy does not account for this
- We can construct accurate Finite Difference stencils with arbitrarily small Dispersion Error

For more information, see Linders, Nordström, Journal of Computational Physics, 2015

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